Transcendence, Truth and Reality: A Mathematician's Fumble

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In which we continue some of the ideas put forward in *Mathematical Sehnsucht* in the first edition of Blue Flower to examine how we understand things that are transcendental to our reality using mathematical concepts and ideas and apply them to our understanding of God.

Introduction

We live in an age in which the existence of objective truth is disputed in order to give some intellectual status to the subjective control of one's identity by the individual. We hear, sometimes in reputable academic establishments, that "my truth is not the same as your truth" and this is to be accepted at the risk of great offence and/or social exile. To assert an objective truth is seen by some as an act of violence upon an individual's existence. Certainly, philosophers such as, notably, Immanuel Kant have separated the impressions of things from things-in-themselves suggesting that the thingness of things is something that can never be known by the senses. If we go that step further, as many modern philosophers and pseudo-philosophers do, and dispute the existence of things-in-themselves then we find ourselves only with the truth of sensory impressions leading to the triumph of the subjective over the objective, and thus the Marxist entitlement of the individual oppressed by a procrustean, objective Society.

There is an alternative, and one that has been known both to mathematicians and also to theologians. If the truth cannot be fully grasped by individual minds, then maybe it exists as something transcendental lying beyond the grasp of sense but apprehensive in part by the rational intellect. This requires careful definition and thought and it is to be admitted that the author is merely a poor student of basic philosophy and not its master, unlike many of his colleagues. The purpose of this essay is to examine the relationship of our understanding of what is truth and our understanding of what is transcendent. We intend to use ideas from mathematics to begin a point of exploration and perhaps to stimulate some thoughts in the reader whose better learning and better grasp of philosophical thought may be more productive and bring about a clearer idea of what is the case.

Transcendence in Mathematics

A Square meal

All things in mathematics begin with a simple idea. A farmer has a perfectly square field of area 9 square miles. If he wants to put a fence all around the perimeter of his field, how long must the fence be?

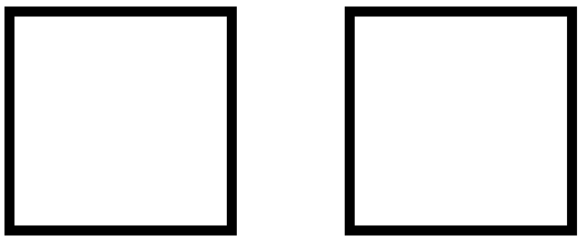
Our thinking will be along the following lines. To find the length of the perimeter, we need to find the length of each side of the square. The only hard fact we know is that the area of the square is 9 square miles. However, we also know that to find the area of a square we have to multiply the length of its side by itself. The question now boils down to finding a number which, when multiplied by itself, gives us 9. That number is 3.[i]

This is why the process of multiplying a number by itself is called squaring. Thus, the square of 2 is 4; the square of 5 is 25; and the square of two fifths is four twenty-fifths. To find the side of a square with a given area is to find the square root. A square of area 25 square miles has side 5 miles; a square of area 4 square inches has side 2 inches; a square of area four twenty-fifths of a square furlong has side two fifths of a furlong.

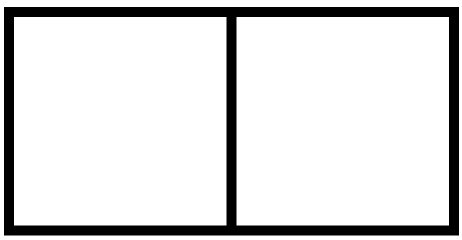
What happens if I have a square of area 2 square feet? What's the side length of this square?

Well, this side length can't actually be a whole number of feet, nor can it actually be a fraction of a number of feet.[ii] Can a number exist that is neither whole nor a fraction? The existence of such a number depends on whether we can construct a square of area 2 square units.

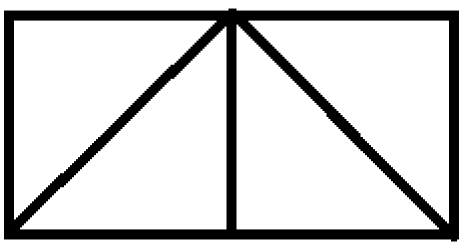
Take two squares of side length 1 unit. They each have area 1 square unit.



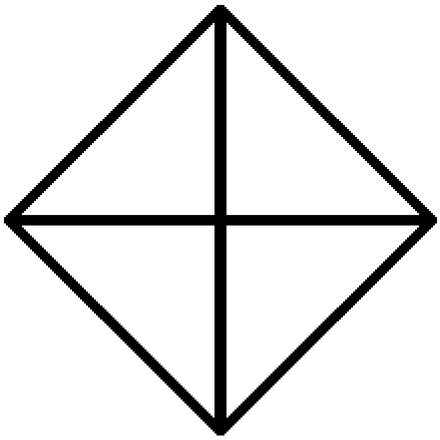
Put them together to create a 2 x 1 rectangle of area 2 square units.



Now divide each square along its diagonal as shown.



Now take the upper two triangles and put them below.



Now, the diagonals of both squares are the same length and meet the base at 45° thus the figure we have constructed has four equal sides and four equal angles: it is a square. We have constructed its area to be two square units. This means that the length of the side of this new square must be the square root of two which we write as $\sqrt{2}$. Thus we have visible, constructible evidence that there is a number which is neither whole, nor a fraction. We cannot even express it completely as a decimal number.

$$\sqrt{2} = 1.4142135623730950488016887242097\ldots$$

This decimal never ends and never repeats itself.[iii]

Cubes, Duels and Ascending Dimensions

What we have done is to give a physical context to the solution of the equation

$$x^2 = 2.$$

This, perhaps, is the simplest example of an algebraic number which is neither whole nor fraction. It is said that the Pythagoreans, infuriated with the existence of such a number, drowned the poor soul (said to be Hippasus) who proved its existence to them.[iv] We can indeed construct the square root of any number we like. While we may not be able to express them in decimal form, we can understand them as numbers get an idea of their size and manipulate them in the way that we manipulate any number whose value we know. The only problem that we find is that we have to refer to them as solutions of an equation rather than by their value.[v]

We have shown that we can double the square and essentially construct the square root of 2 geometrically. Can we double the cube in the same sort of way?

What does that mean? Can we construct a cube of volume 2 cubic feet from two cubes of 1 cubic foot each in the same sort of dissection as we did for the square?

Algebraically, we could say that we seek a way of finding a number which when I multiply it by itself twice, I get 2, i.e. find a number x so that

$$x^3 = 2.$$

This equation defied construction for a long, long time until Evariste Galois, the night before he was to be shot in a dual, set up the mathematical framework to show that it isn't possible. We can't dissect two cubes to construct another in the same geometric way as we did for squares. To do so requires special techniques not available to the Greeks who were armed only with a straight edge (without markings) and a pair of compasses. Nonetheless, it's clear that such a number must exist as we can certainly think about how we can do this with pieces of plasticine (which wasn't available to the Greeks!).

We can understand, therefore, that there are a whole myriad of numbers which solve equations which involve powers. These are called algebraic numbers and they all exist as solutions of equations of the form:

$$a + bx + cx^2 + \cdots + x^n = 0$$

where the numbers *a,b,c,* et c, are either whole numbers or fractions. The highest power (n in this case) is called the degree of the equation. Thus, the problem of finding the side of a square given its area is a degree 2 problem. The problem of doubling the cube is a degree 3 problem. The upshot is that there are suddenly a whole plethora of new numbers that are neither whole numbers nor fractions which must extend our thinking beyond our ideas of finite decimals.

Does this mean that we have all the numbers now?

Having our pi and e-ting it.

Actually, we haven't even scratched the surface. There are more numbers that exist than are algebraic. These are the famous transcendental numbers and most of us are already familiar with at least one of them.

The ratio of the circumference of a circle to its diameter is the number π. We often approximate this number by the fraction $^{22}/_{7}$ but its true value is much more complicated. Again, we find that it is neither a whole number nor a fraction. The question is, is it algebraic? Does it solve an equation of degree n where n is a finite number?

The answer is no, and this is demonstrated by the Lindeman-Weierstraß theorem – a result formed from the work of two nineteenth century mathematicians. In fact, we can find π as the solution of an equation of *infinite* degree but algebraic numbers solve equations with finite degree. We know that π is finite and yet its very existence is bound up with all kinds of infinities within its finitude. The same is true of another mathematical number e which despite its intractable definition keeps cropping up again and again in Mathematics, Physics, Biology and Economics. It seems that we cannot adequately describe our physical reality without numbers that go beyond our grasp.

What we can do, however, is approximate.

The mathematical waving of hands

In practice, we do not need to know a number infinitely precisely. Measuring a field's perimeter, we can be satisfied with the nearest yard, foot or inch depending on the circumstances and depending on how picky we are. We are not able to get absolute precision: indeed, according to Quantum Mechanics, we are never physically able to measure distances smaller than 0.00000000000000000000000000000000063 inches. In practice, we don't even need that. We now know π to over a billion (UK) digits, yet to calculate the circumference of the galaxy, we would only ever need to know it to 32 digits.

However, mathematics goes beyond just practice. In mathematics, we have methods that allow us to get as close as we like to a number. Numerical methods allow us to write computer programmes to approximate numbers like e, π and $\sqrt{2}$ to any degree of accuracy that we might require.

For example,

Step 1: Pick a number.

Step 2: Add 1 to that number.

Step 3: Divide the result into 1.

Step 4: Add 1 to the result.

Step 5: Go to Step 2

After sufficient repetitions, we find ourselves getting numbers at the end of Step 4 which are arbitrarily close to $\sqrt{2}$.

We can do the same for all numbers, algebraic or transcendental. Essentially, we can show that the number line that we drew at school just to add and subtract is not just comprised of whole numbers and a few select fractions, but is a complete continuum – an unbroken line in which whole numbers and fractions are the real oddities, the algebraic numbers a little more common, but the transcendental numbers the most frequent examples of what it is to be a number.

The Mathematician can wave her hands in writing down a transcendental number to only a few digits, but she knows that that number is unique and exists as a unique yet infinite decimal.

For the mathematician, the essence of Transcendence is infinity within the palms of our hands and Eternity in an hour.[vi] Physics brings to our mind that our existence is clumpy, finite and rarified; Mathematics brings to our mind a seamless existence with no gaps and that behind the clumpy physical existence there are precious things to be discovered that physics cannot possibly reach.

Transcendental truth

Is there an objective truth? Well, what might that mean? A real fact about our existence that everyone can and must agree on? A perfect correspondence between "out there" or in the mind? Logic would dictate that objective truth must exist.[vii] We can also say, quite objectively, "something exists." The question then becomes "can we access objective truth?" – that is a different problem and one that continues to be examined even today. Can we take heart from Mathematics?

MPP and the Monkey's Uncle

The truths of mathematics are largely objective if one accepts the standard logical rules of inference, especially *modus ponens ponendo* (MPP). This is the method of logical inference, i.e. if P is true means that Q is true, then establishing P's truth establishes the truth of Q or, more formally:

1) P
2) If P then Q

Therefore

3) Q

Yet does this work? Is it objectively held?

At present, there seems only to be one objection to the validity of MPP. Let's consider the following argument (McGee, 1985)

1) Either John Cage or Beethoven composed the Eroica Symphony.
2) If either John Cage or Beethoven composed the Eroica Symphony then, if it wasn't Beethoven, John Cage composed the Eroica Symphony.

Therefore

3) If Beethoven didn't compose the Eroica Symphony then John Cage did.

Of course, the apparent problem lies in the fact that one can substitute any name for John Cage, such as Philip Glass for example, and the argument remains true. Also, if Beethoven didn't write the Eroica Symphony, then there are more plausible composers who might have – Mozart, for example.

If we put this argument into more abstract terms, we find:

1) Either P or Q[viii].
2) If either P or Q then, if not P, Q.

Therefore

3) If not P then Q.

Logically this is valid. In fact, statement (2) is a tautology because

If not P then Q

is the same thing as saying

Either P or Q.

Thus McGee's argument becomes a logical tautology.

What McGee is trying to do is to cast doubt on the validity of MPP. The basic definition of validity namely that if the premises of an argument are true then the conclusion must be true. McGee points out that he has given an instance of MPP in which the premises are true but the conclusion is not obviously true. The trouble is that he is using language which could be considered imprecise or equivocal. The fact of the matter is that it is perfectly correct to say that if Beethoven didn't compose the Eroica then John Cage did. It's exactly the same as saying, "if Beethoven didn't compose the Eroica then I'm a monkey's uncle," which in turn is logically equivalent to "If I'm not a monkey's uncle then Beethoven composed the Eroica." This is known as the contrapositive statement: the contrapositive of

If P then Q

is

If not Q then not P.[ix]

McGee has transplanted the question of the validity of MPP from being a question in Logic to a question in Language, and herein lies the problem with much of Postmodern and relativistic thinking.

MPP is a crucial part of logic and reason that we use to draw conclusions, and its failure would have significant disruption on the way we communicate and argue. It would certainly introduce doubt on the system and it would also throw into doubt the same reasoning that enters into every culture. However, we have shown that McGee's logic reverts actually to an idiom that we use in everyday life – the Monkey's Uncle.

What McGee believes that he has found is an argument against our whole system of inference which calls our trust in logic into question. However, his appeal is not to Logic itself but rather to the way we understand phrases and give meaning to conclusions. Even with the introduction of Fuzzy Logic in which there is a sliding scale of veracity between Absolute Truth and Absolute Falsehood, Classical Logic presents possibilities for the Absolute that lie outside the confines of the human senses and, since Classical Logic is embedded within Fuzzy Logic, the manipulation of Objective and Absolute Truth is still possible yet only approximated by our empiricism. Transcendence remains present, despite others' attempts to deny it.

Playing the Peano

The question "Can we access objective truth?" has a relatively simple answer. We cannot access **all** objective truths. This can be proved perfectly logically as Kurt Gödel demonstrates in his incompleteness theorems. Essentially, there are statements in mathematics that cannot be proved true or false and, further, we cannot know which statements these are.

For example, we might look at the famous Goldbach Conjecture that every even number greater than 4 can be expressed as the sum of two primes. This is still unproved after 276 years and the question remains whether it is in fact provable or an example of Gödel's unprovable statements. Given that it took 350 years to prove Fermat's Last Theorem,[x] it is clearly worth continuing the search if only to discover greater and more powerful mathematical theories and methods.

But can we access **any** objective truths? Clearly basic arithmetic provides this objectivity. It doesn't matter what language we use, the truths of arithmetic remain. If we can perceive an object with our senses then imagining a copy of it is perfectly reasonable, and clearly the process is repeatable.

We can follow Descartes here quite nicely, especially with his *cogito ergo sum*. If we can perceive nothing with our senses, then we can think of our mind considering nothing that we can perceive. Since this will get overly verbose if we let it, we'd better introduce notation. If we are considering C in our mind, let us write $\{C\}$ - the braces effectively standing for our mind. Let us write $\emptyset$ to stand for our mind considering the nothing we can perceive, i.e $\emptyset = \{\quad\}$.

If we can now think of our mind considering nothing, we now have $\{\emptyset\}$, and if we consider thinking about our mind considering nothing we have $\{\{\emptyset\}\}$.

Now although this could get very convoluted in words, we can generate a whole load of new thoughts for, if we can consider C then we can also consider C along with our mind considering nothing. That is, if $\{C\}$ then $\{\emptyset, C\}$. This gives us a process of generating new thoughts.

$$\begin{gathered}\emptyset = \{\quad\} \\ \{\emptyset\} \\ \{\emptyset, \{\emptyset\}\} \\ \left\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\right\} \\ \vdots\end{gathered}$$

Although this might take some mental gymnastics of which the trapeze artists at Barnum and Bailey's might balk, what we have just done is set up a system that was formalized by Giuseppe Peano in the latter half of the 19th Century.

First, he explains what properties "=" has.

1) For any x, x=x;
2) If x=y then y=x;
3) If x=y and y=z then x=z;
4) If x is in C and x=y then y is in C.

These are what we expect the properties of identity to have. Following McGee's caveat, we do have to be careful with language because if we say "God is Love" then can we also say "Love is

God"? That requires greater precision with definitions especially when "Love" is largely an equivocal term and that very equivocation is responsible for the moral decline of mainstream Christianity. If, however, we are restricting our attention to people in the phonebook and we know that Bell is Wake and that Wake is Atkinson, then we know Wake is Bell, Bell is Atkinson and that if Bell is a person, then so are both Wake and Atkinson.

Peano then goes on. He takes as his starting point the number 0 and a process on numbers called S which is called the successor process. If we apply the process S to a number x, we write $S(x)$.

5) If x is a whole number, then so is $S(x)$.
6) If x and y are positive whole numbers, then $x=y$ if and only if $S(x)=S(y)$.
7) There is no positive whole number x so that $S(x)=0$.
8) If 0 is in C and $S(x)$ is in C whenever x is in C, then every positive whole number is in C.

This last and rather involved idea is called the Inductive Principle and can be illustrated happily by the action of climbing a ladder. If we know that if we are on the rung of a ladder, then we know we can climb onto the next rung. So, if we find ourselves on the first rung of a ladder, we know we can climb up as far as we like (or until we run out of rungs). If C consists of only the rungs we can climb, then our first rung is 0 and is in C, and if we are on rung x then we know we can climb up to the next rung $S(x)$, so if x is in C, then so is $S(x)$. We can climb all the rungs of the ladder so, all the rungs of the ladder lie in C.

From these ideas, Peano builds up all of our arithmetic. For he can say that $1=S(0)$, $2=S(1)$, and so on. All he has done is what every toddler does: he has learned to count. Of course, he has just put that into more rigorous a framework. Essentially, just knowing 0 and S gives us all the positive whole numbers and, from these, we can obtain all numbers - positive, negative, fraction, algebraic, imaginary, transcendental...

If, however, we go back to thinking of nothing, we find that we can start with $\emptyset$ and then say that $S(C) = \{\emptyset, C\}$ and find that[xi]

$$\emptyset = \{\ \ \} = 0$$
$$\{\emptyset\} = S(0) = 1$$
$$\{\emptyset, \{\emptyset\}\} = S(1) = 2$$
$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = S(2) = 3$$

and so on.

In thinking of nothing, we can still build up numbers quite naturally and objectively. We literally bring about something out of nothing – creation *ex nihilo*! Nothing physical has to exist for this to happen – indeed, nothing physical can be brought into existence from this process. This does demonstrate that our search for the truly objective is something that transcends our physical existence. We cannot expect to find the Absolute within the confines of matter which is only finitely divisible. Again, we find Transcendence within the very act of thinking.

Derrida: Deconstruction or Fractal?

Jacques Derrida would bid us to seek the truth by taking up the contrary position to our own. Famously he seeks to redress a balance from having the search for truth which is often expressed in writing to be expressed in other means. To be faithful to Derrida, we need to be able to find the opposite to Metaphysics, Logic and Reason, the Logocentric Tradition as he would call it. For him, the Western Tradition has preferred the written word over the spoken word, and words over pictures, and these preferences are without foundation. He sees the dualities of “true” and “false”, “male” and “female” as being unnecessary complications which arise from Logocentrism and that we are overly-concerned with this binary segregation.

The trouble is, in order to examine Derrida’s ideas, we necessarily have to rid ourselves of both Reason and Anti-Reason in order to make a proper deconstruction. Is this possible? Can human beings truly get beyond every thesis-antithesis duality in order to find the Truth? This sounds very much as if Derrida is trying to out-Hegel Hegel. Yet, Derrida is objecting to the very necessity of the thesis-antithesis duality itself.

In searching for the anti-reason that would allow us, in Derrida’s eyes, to get a clearer picture of what is True, we could very much look at Jung’s psychology (Jung, 1971) and its bringing to birth of the Myers-Briggs theory of personality typing (Myers & Myers, 1995 (1980)). Reason and Anti-Reason are encoded in the personality in the way we make our decisions. For example, according to Myers and Myers-Briggs, the judging faculties of the individual lie on a sliding scale between Thinking and Feeling. We make decisions based upon our reasoning faculties or upon our “gut”. It seems then that, in order to appreciate Derrida’s deconstruction, we need to seek truth with our feelings.

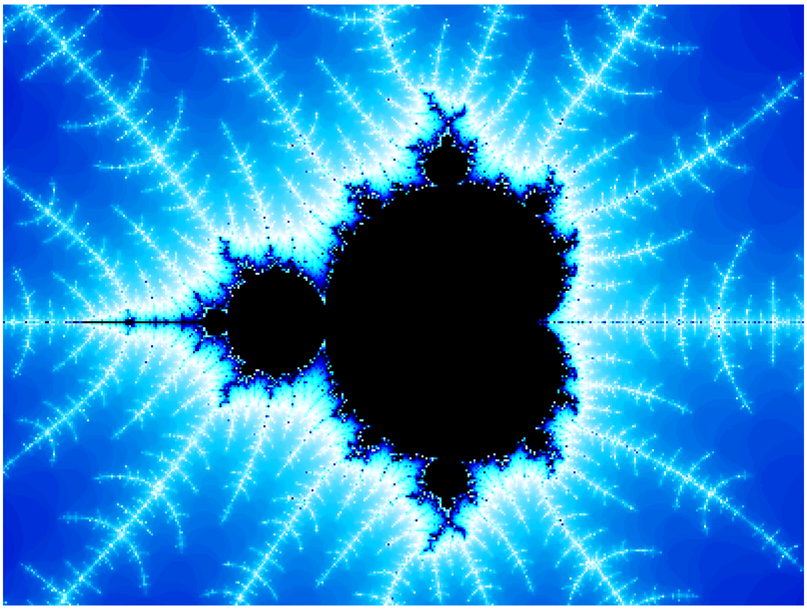
For Derrida, this is the importance of art in Society: music, painting, sculpture and the like are as much vehicles to what is true as the search for knowledge through reasoning and argumentation. He might be horrified by the compartmentalism of Jung-Myers-Briggs psychology, and there would be good reason as the Human Spirit seeks to be unbound by such a classification into dichotomies. If Derrida sees such binary polarity as unnecessary, it will be in the way that psychology believes that it can understand human behaviour.

One thing that Gödel’s theorems of Incompleteness show is that it is impossible to obtain all truth within any axiomatic system. Derrida would be overjoyed with this. Logic itself is incomplete in its scope

We have seen that we can construct $\sqrt{2}$ algebraically (as the solution of the equation $x^2 = 2$), practically (as the length of the side of a square with area two units), algorithmically (via the repetitive formula given above) and geometrically (via the dissection of the square in pictures) without ever explicitly writing the number down. What might Derrida make of that?

It is conceivable that Derrida might appreciate the beauty of the construction whereby we use manipulation of what is essentially concrete and graspable to construct explicitly, visually and coherently a number which cannot be noted down save only by the notation $\sqrt{2}$. And that is something where Thinking meets Feeling, for it is within the geometry of systems that we can generate ideas, figures, and concepts which touch us at the intersection of thought and emotion.

One such intersection that became fashionable in the 1980s is the notion of fractal – images which possess structure on every scale of magnitude. Many videos exist of zooming into the Mandelbrot Set[xii] which demonstrate this and can produce such emotions of euphoria that the images have been used in the Art world as well as in the “rave” culture of the 1990s.



Yet such is the simplicity with which fractals can be generated, that we can touch upon natural phenomena. A process of iteration, now developed into a highly sophisticated method within computer graphics will produce the image of a fern.

We are now at sufficiently sophisticated a technology that, conceivably, we can ask ourselves, “is this real Romanesco Cauliflower, or is it a computer image generated by fractals?”

As Douglas Hofstater takes pains to explain (Hofstater, 1999), the music of Bach with its self-referential nature and almost fractal complexity is as much a testament to the humanity of mathematics as it is the mathematics of humanity. Perhaps this is the passion for truth in its rational complexity and transcendence, found within the breast of the Mathematician, that Derrida seeks set against logic that gives it a greater access to what is true even if that truth cannot be grasped in full.

Ravening Pythagorean Wolves

We have already seen the intimate relationship between algebra and geometry to give us insights into finding numbers and then back again. We could even involve music, which Pythagoras regards as a branch of mathematics, to illustrate the existence of irrational numbers.

Just from observing the behaviour of a plucked string, the Pythagoreans were able to deduce that halving the length of a string doubled the frequency of the pitch of the note and produced a note an octave higher than the open string. Likewise a string of a quarter of the length quadruples the original frequency and thus produces a note two octaves higher and an eighth of the length octuples the original frequency and produces a note three octaves higher, et c.

However, a string a third of the original length trebles the frequency but produces a note an octave and a perfect fifth above the original. This tells us that we can construct a perfect fifth above the original by a string of two thirds the original length.

Now, as any musician will demonstrate, the process of successively finding the perfect fifth above will construct, in theory, all twelve notes in the octave and finally arrive back at the octave. However, it doesn't! The theory is wrong. Why?

To find the perfect fifth above a note, we multiply its frequency by $^3/_2$. Do this twelve times and we should end up seven octaves above where we started.

This means we should have the equation

$$\left(\frac{3}{2}\right)^{12} = 2^7.$$

However, this is untrue for

$$2^7 = 128$$

and

$$\left(\frac{3}{2}\right)^{12} = 129.746337890625\ldots$$

The discrepancy is called the Pythagorean comma which, on early keyboard instruments such as the harpsichord, virginals and spinet, there occurs the wolf note rendering some keys unplayable.

However, as John Bull's remarkable Fantasia on Ut Re Mi Fa Sol La[xiii] shows, one can take a different approach to tuning the virginals. If we require twelve notes to the octave and that the octave higher has twice the original frequency, then we must need to construct $\sqrt[12]{2}$, i.e. the positive number which solves the equation,

$$x^{12} = 2.$$

Twelve Pythagorean perfect fifths are not the same as seven perfect octaves and the comma occurs because we seek to approximate this irrational number by fractions. Other approximations have been found in antiquity. Indeed, the Chinese discovered that the comma between the thirty-first octave and the fifty-third perfect fifth is smaller than the comma between seven octaves and twelve perfect fifths. If we were to continue indefinitely, we may gain better and better approximations.[xiv] Of course, there is only so far our hearing can distinguish the resulting commas.

Nonetheless, just like our ideas of approximation, we can find practical ways of improving our tuning almost naturally as clearly occurred in the Sixteenth Century before Galois, Lindemann and Weierstraß and the whole theory of algebraic numbers were rigorously treated in mathematics and certainly before electronic tuners appeared.

The Absolute and God: towards Mathematical Spirituality

Although he would reject the idea of the existence of God, Derrida ought to appreciate the lengths at which the Church has gone to present the truth of Christian belief. The truth within Holy Scripture is grasped not just by the literal:

Lettera gesta docet,
quid credas allegoria,
moralis quid agas,
quo tendas anagogia.[xv]

The Seventh Oecumenical Council also rather demands that we learn to see beyond physical appearances through the use of ikonography.

From this point of view, Christian spirituality is simply not logocentric even though its object of worship is the Divine *Logos*. The painting of medieval churches sought not just to teach but inspire the faithful gathering for worship and also to allow the craftsman to express his worship through the work of his hands. While the Protestant Reformation largely robbed the faith of some its colour, Catholicism has retained both rational and artistic dimensions to its indication of the truth of Christian Doctrine. This might explain why the majority of the world's population are Roman Catholic. Where does God, the Absolute Transcendental fit in to all this?

When it comes to believing in God, we have a choice to make and this is the same choice that we are faced with when we consider numbers such as $\sqrt{2}$. Physically, it is impossible to draw the perfect square: mathematical lines have no width unlike the marks we make with a pencil; perfect right angles require perfect pairs of compasses. If we cannot construct the perfect square, then we cannot construct $\sqrt{2}$ perfectly. We can posit the idea of ideal lines and we find ourselves firmly in the realm of Platonic Idealism. Similarly, we run out of octaves to return to unison with the cycle of perfect fifths.

In all these examples, there is the sense that we are closing in on some number, some element, some being that exists forever out of reach and must always remain outside our universe and experience. The practical materialist will say that the reality of these transcendental limits is irrelevant and we might as well treat them as being figments and constructs of the mind with no reality external to it. The fact, however, is different. Algebraic numbers and transcendental numbers behave like numbers in precisely the same manner as whole numbers and fractions. We can make the same objective and quantifiable statements as we might whole numbers. We may exist in a sort of Zeno's paradox where we cannot leave the room by traversing half the distance to the door each time, but that does not demonstrate the non-existence of transcendentals. They leave their mark on reality through infinite decimals, circumferences of circles, ever finer gradations of coast-lines, ascending octaves, and endless computer algorithms. If we exclude the possibility of their being out of hand, then we miss vital evidence of the richness, not just of the empirical world, but also of the possibility of Kant's noumena.

Timothy Williamson (Williamson, 2002) posits the idea that Knowledge is itself basic and therefore cannot reliably be analysed into component parts of true, justified belief plus extra conditions. Rather, Belief itself is an imperfect form of Knowledge or, perhaps in Platonic framework, Knowledge is the form of Belief. Williamson's idea presents perfect Knowledge as something that is transcendental to the physical realm. Thus in its use of probability in which to draw its startling conclusions, Quantum Mechanics admits that it cannot know things for certain: particles could be subject to the principles of chaos at the smallest levels. Probability is the mathematics of "fudge".

But what about the Absolute? Can that exist?

It appears that our attempts to understand the Transcendent would suggest that it does, though we cannot have knowledge of it. We can understand the idea of Absolute Infinity in mathematics by use of the Reflection Principle. Naively, this says, for any property of the universe of all sets we can find a set with the same property. There is an inherent contradiction in this,[xvi] but there are ways of shoring the argument up to produce a coherent Reflection Principle which is very useful for us.

> *"… this means that we do, in fact, know something about the Absolutely Infinite: all of the properties it possesses must be shared with and disclosed to us through the properties of the transfinites. The Absolute Infinite is in this sense knowable, comprehensible; each*

of its properties must be found in at least one transfinite number. The Absolute is disclosed through the relative, Absolute Infinity through the transfinite, and yet it is precisely through this same disclosure that Absolute Infinity remains hidden, ineffable, incomprehensible . . . it is as though the transfinites form an endless veil surrounding Absolute Infinity. The veil is all we can ever know . . . Yet, genuine knowledge about Absolute Infinity is forever revealed in the veil that hides it. . . ." [xvii]

And we also begin to appreciate the work of the Early Theologians in their understanding of God and His attributes. We begin to understand God by our encounter with Him and how our actions can reflect His, though in a limited way. The fact of the Incarnation in Jesus Christ shows that God can be approached in human terms before our understanding wavers. We cannot understand the concept of the Trinity because there is no human experience of having more than one locus of identity.[xviii] Even in the Person of God the Father, the Old Testament speaks of His personality though we recognize it to be a Platonic Reflection or Shadow of His true personality.

Conclusion

In beginning with simple problems that humanity has faced, we have found the existence of numbers that go beyond our reckoning and even beyond the solution of equations. We have seen mathematics provides the notion not only of limits, but of how we can approach and understand those limits. Passing from pure mathematics, we see that attempts to derail the objectivity of inference fail because of the very nature of being able to infer from given premises and thus mathematics retains its objectivity. We have seen how thought itself generates not only objective truth but also how the truth that it builds becomes transcendent. We have shown that the truths that mathematics raises can be met in opposing methods of communication, in art, music, and passion versus logic, word and reason.

In all cases, the question of how we deal with transcendental realities is to use the best version of the Reflection Principle that we have. We begin by what we know and adjust and approximate as best we can, at each step refining our reasoning and experience.

Our search for God lies within the world around us as we approach the limits of what we can experience of the physical world and of other people, within Holy Scripture as we approach the limits of God's revelation to human beings, and within our own consciousness as we approach the limits of what we really know. Ultimately, the best use of the Reflection Principle is the life of prayer itself, for prayer prevents us from treating God as an abstract object but rather as One with Whom one can communicate. If the atheist seeks to prove the non-existence of God, then he must assume that God exists and show that the Reflection Principle fails for everyone, for if one person truly finds God, that is proof enough of His existence. The trouble is that every other person is transcendental to us, otherwise we would know what it is to be them.

It seems that Christians have been applying the Reflection Principle for centuries before Cantor. Maybe this is evidence for its own transcendence.

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[i] Mathematically, the answer could be -3. However, in the context of lengths of fields, this is not a meaningful answer. If we were talking about a temperature which squares to 9, then we could potentially have two answers, 3 and -3.

[ii] The proof of this is a classic *reductio ad absurdum* argument which it is better to reserve to these notes and included for the benefit of the interested reader. It runs as follows.

By definition, a fraction is the ratio of two whole numbers. So, we make the assumption that there is a fraction $\frac{p}{q}$ where p and q are whole numbers which have no common factors and which squares to 2. This means that

$$2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2}$$

By multiplying both sides of this by q^2, we can see that

$$p^2 = 2q^2.$$

This means that p^2 is an even number because it is twice another whole number, namely q^2. This must mean that p itself is an even number, because an odd number squared must be odd. This means that

$$p = 2k$$

for some other whole number k. In which case.

$$p^2 = (2k)^2 = 4k^2.$$

This means that

$$4k^2 = p^2 = 2q^2$$

in which case

$$2k^2 = q^2.$$

However, this means that q^2 is an even number. This would mean that q itself must be even. We have already established that p is even so now both p and q are both even numbers. This means that they have a common factor of 2. However, as p and q were chosen to have no common factors, we have a contradiction. This means that if there is a fraction that squares to 2, then two numbers which have no common factors actually have common factors. Since two numbers which have no common factors cannot have common factors, there can be no fraction that squares to 2.

It's worth noting that we have done something that many people think is not possible: we have proved a negative statement.

[iii] i.e. it doesn't repeat the same string of numbers indefinitely such as $^1/_3 = 0.33333333 \ldots$, $^3/_7 = 0.428571428571428 \ldots$ or $^7/_{600} = 0.01166666 \ldots$ All fractions have decimals that eventually terminate or repeat like this.

[iv] Kurt von Fritz, "The discovery of incommensurability by Hippasus of Metapontum", Annals of Mathematics, 1945.

[v] Actually, we've been doing that for a long time. When we write $\frac{1}{2}$, we are referring to the number x that solves the equation

$$2x = 1.$$

Likewise, $\frac{p}{q}$ is the number x that solves the equation

$$qx = p.$$

[vi] To paraphrase William Blake.

[vii] Another *reductio ad absurdum* argument displays why this has to be the case. Suppose there is no objective truth. Then "there is no objective truth" is something we would all agree on by the hypothesis. However, "there is no objective truth" becomes an objective truth which is a contradiction.

[viii] In logic "Either P or Q" also allows for the possibility that both P and Q are true. It's not an exclusive "or".

[ix] It's worth remembering that the famous false inference comes from making the leap from

If P then Q

to

If not P then not Q.

This is very easily seen with P being the statement "Tiddles is a cat" and Q the statement "Tiddles has four legs".

[x] That there are no whole numbers a, b, c for which $a^n + b^n = c^n$ for some whole number n bigger than 2. Although he claimed to have proved it in the margin of his book, Fermat almost certainly did not have an adequate proof.

[xi] In case there is a lack of clarity, it's important to know that $\{\emptyset, \emptyset\} = \{\emptyset\}$. This is because there is only one thing in $\{\emptyset, \emptyset\}$, namely $\emptyset$.

[xii] See, for example https://youtu.be/PD2XgQOyCCk, only find your most favourite piece of music to listen to for seventeen minutes as you watch.

[xiii] Fitzwilliam Virginal Book, Number LI

[xiv] One such is 665 perfect fifths to 389 octaves! Now try finding a piano to play that!

[xv] Paragraph 118 of the Roman Catholic Catechism (Vatican, 2002)

[xvi] See Mathematical Sehnsucht in the first edition of Blue Flower.

[xvii] Robert John Russell in Chapter 13: God and Infinity: Theological Insights from Cantor's Mathematics, (Heller & Woodlin, 2011)

[xviii] Mathematically, we can indeed understand from examining the concept of the Universal Set that it is conceptually possible for a One to be regarded as a Many, and a Many as a One. Again, see Mathematical Sehnsucht in the first edition of Blue Flower.